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The authors have studied a novel numerical solution approach to Monge-Ampere type equations. The equations have important applications. In particular, the leading term in the "balance equation" in dynamic meteorology has the form (1). In a more complicated form, an equation of this type appears in the von Karman system of equations for elasticity and also in inverse problems of geometric optics. It also turned out that recent progress in the study of fully nonlinear equations became possible after important properties of the equation (1) were discovered.

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NEW METHODS FOR NUMERICAL SOLUTION OF ONE CLASS OF STRONGLY NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS

Final Technical Report
AFOSR Grant 87-0140

V.I. Oliker
and
P. Waltman

During the previous and current funding periods by the AFOSR the following two main areas I and II of research were pursued:

Below we describe progress made on these topics.

Part A: **General Remarks**

The physical phenomena described by nonlinear partial differential equations have become at present the central theme of investigations by many researchers. A good understanding of most physical processes requires accounting for nonlinear effects and, consequently, methods for studying nonlinear equations have to be developed.

Among nonlinear equations the Dirichlet problem for the Monge-Ampere equation is the model case for fully nonlinear equations. The problem is formulated as follows.

In Euclidean plane R^2 with Cartesian coordinates x,y consider a bounded domain Ω, a nonnegative function $f : \Omega \to [0,\infty)$, and a continuous function ϕ: $\partial\Omega \to R$. It is required to investigate solubility of the problem

$$M(z) \equiv \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = f \text{ in } \Omega, \tag{1}$$

$$z\,|_{\partial\Omega} = \phi \tag{2}$$

The equation (1) is a representative of the class of nonlinear equations of Monge-Ampere type. Such equations have been studied by many authors, mainly in connection with problems of existence and uniqueness of surfaces with prescribed metric or curvature functions. However, they also have other

important applications. In particular, the leading term in the "balance equation" in dynamic meteorology has the form (1). In a more complicated form, an equation of this type appears in the von Karman system of equations for elasticity and also in inverse problems of geometric optics. It also turned out that recent progress in the study of fully nonlinear equations became possible after important properties of the equation (1) were discovered. For quasilinear elliptic and parabolic equations the use of equations of the form (1) is crucial in obtaining C^0 and Hölder estimates.

Part B: **Numerical solution of equations of Monge-Ampere type**

In view of the important practical applications several heuristic approaches were suggested earlier for numerical solution of some modified forms of (1), (2). Though no rigorous analysis of these methods exists, one may note that they all are local methods based on a finite difference approximation and linearization. Because of the strong nonlinearity of M this approach might be successful only in a neighborhood of the true solution and therefore, if a priori good initial approximation is not available, these methods will not produce, generally, a sequence converging to the true solution.

We have investigated this problem in detail and obtained the following results:

- A special discretization scheme for (1), (2) was suggested, different from standard finite element or finite difference schemes. It can be shown that latter ones in known forms will not work here.
- an iterative method has been developed for solving the discretized version of (1), (2);
- the question of finding an initial approximation in our scheme is completely and effectively resolved; it is just a routine step of the iteration process;
- the iterations are self-correcting;
- global convergence is established;
- our algorithm is suitable for a parallel computer;
- a computer code has been written and tested on a serial and parallel machines.
- a graphics package accompanying the code was also developed.

The experience gained in testing our procedure on different types of examples, including ones with large gradients, shows that its most effective use is in combination with some fast Newton-type scheme. Consequently, a particular criteria is established for checking when the current approximation can be used as the beginning step for a converging Newton-type iterative procedure.

- In this combined scheme we proved convergence (quadratic) of the Newton iterates.

The computer code for the method is quite sophisticated; it involves, as a step, construction of a convex hull of sets of points in R^3. There are here different aproaches and the effectiveness of the algorithm depends on it. We have tested various schemes and the code presently is a substantial improvement over its original version of 1984-85. Recently our work here was followed by Baldes and Wohlrab in Germany.

Part C: **Numerical solution of the "balance equation" arising in weather forecasting**

We developed extensions of our numerical procedure to elliptic equations of the form

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2+a(x,y)\frac{\partial^2 z}{\partial x^2}+2b(x,y)\frac{\partial^2 z}{\partial x \partial y} \tag{3}$$
$$+c(x,y)\frac{\partial^2 z}{\partial y^2}+g(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y})=0.$$

A special case of this equation is the "balance" equation

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}-\left[\frac{\partial^2 z}{\partial x \partial y}\right]^2+\operatorname{div}(p(x)\operatorname{grad} z)=f \tag{4}$$

to be solved in a convex bounded domain $\Omega \subset \mathbf{R}^2$ subject to the boundary condition

$$z|_{\partial\Omega}=\phi. \tag{5}$$

where ϕ is a continuous function.

This equation relates the rotational component of the wind velocity and the geopotential. It is used in a model of numerical weather prediction.

For the problem (4), (5) we define a notion of a generalized solution by replacing (4), (5) by an equation in measures

$$\int_\omega\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}-\left[\frac{\partial^2 z}{\partial x \partial y}\right]^2\right)dxdy+\int_\omega \operatorname{div}(p(x)\operatorname{grad} z)\,dxdy=\int_\omega f\,dxdy$$

where ω is an arbitrary subdomain of Ω with piecewise smooth boundary. The first term on the left fits into the scheme developed by us earlier. For the second term we construct a special discretization compatible with the one used for the first term. As a result we obtain a system of quadratic equations. This system is solved by an algorithm which is an extension of the one in part B. A code for this procedure has been written and tested.

Part D: **Evolutions of fronts in flows governed by nonlinear equations ("burning surfaces")**

During the summer of 1988 we started investigations of flows generated as solutions of the equation

$$\frac{\partial z(x,y,t)}{\partial t}=\frac{\partial^2 z(x,y,t)}{\partial x^2}\frac{\partial^2 z(x,y,t)}{\partial y^2}-\left[\frac{\partial^2 z(x,y,t)}{\partial x \partial y}\right]^2-f(x,y,t). \tag{6}$$
$$(x,y)\in\Omega,\ t\in(0,\infty)$$

$$z(x,y,0)=z_o(x,y),\ (x,y)\in\Omega,\ z(x,y,t)|_{\partial\Omega}=\phi(x,y,t),\ t\in(0,\infty), \tag{7}$$

where f, z_0 and ϕ are given functions.

This research is motivated by several facts. First of all, under some assumptions we can show that as $t\to\infty$ the solution $z(x,y,t)$ converges to the solution of the steady state equation

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}-\left[\frac{\partial^2 z}{\partial x \partial y}\right]^2 = f_\infty \, in \, \Omega,$$

where the right hand side $f_\infty = \lim f(x,y,t)$ as $t \to \infty$. This is exactly the equation (1). Thus, we have an alternative method for constructing solutions of (1), (2).

In the numerical computations this approach has the advantage that one does not have to invert the nonlinear operator M but, instead, just computes its value on the current update. The price for that is the need to do a lot more iterates (in time) than in the method for solving (1), (2) as described in part B. However, for domains with complicated geometry, or when the operator on the right hand side of (6) is substantially more complicated than M, this may be an acceptable payoff because each iteration here is fairly simple to perform, and fewer checks along the way are needed.

Another factor motivating the study of (6) stems from the observation that (6) describes a flow with a front moving with curvature dependent speed (the first term on the right of (6) is proportional to the Gauss curvature of the surface front). Similar flows were used in physics to describe combustion models (this is the origin of the term "burning surface"). Flows involving Gauss curvature arise in several applications; for example, the rate of wear of certain materials is proportional to the Gauss curvature.

The equation (6) is also interesting in itself because it represents a flow governed by a genuinely nonlinear differential operator in spatial derivatives.

Many of the ingredients constructed in parts B and C were utilized in developing a discretization scheme for (6). We have written a code for numerical solution of (6) and tested it on a number of examples. This code proved to be very useful in testing various conjectures about flows. On the other hand the numerical experiments that we ran suggested a number of interesting properties of such flows, not all of which have been known even for classical parabolic equations. The results of this investigation are in paper "Evolution of nonparametic surfaces with speed depending on curvature, I. The Gauss curvature case". A special case of such a flow is a front propagating with normal speed proportional to the Gauss curvature of the frontal surface. In this paper sharp estimates of the asymptotic rates of decay are established and these results are applied to show that such flows "pick up" asymptotically the symmetries of the boundary of the domain.

Part E: **Numerical solutions of nonlinear equations and computer graphics.**

Since the iterates in numerical approximations of solutions of (1), (2) are represented as convex hulls of sets of points in R^3, it is natural to represent those iterates as graphs of polyhedral surfaces in R^3. During development of algorithms for numerical solutions of PDE's it is extremely useful to see on the computer screen the surfaces corresponding to each iterate and how they evolve in the process of updating. We could not find appropriate software on the market and developed our own, suitable for our purposes. The software that we developed generates a sequence of surfaces on the screen. Those surfaces are colored depending either on particular physical properties of the solution or on the point of vision of the observer.

Part F: **Applications**

A particular nice practical application of nonlinear equations of Monge-Ampere type is the problem of antenna design. In this problem it is required to determine a reflecting surface such that for a given point-source of light the reflected rays cover a prescribed region ω of the far sphere and the density of the distribution of reflected rays is prescribed in advance as a function of the reflected direction. It is assumed that the power density of the source as well as the aperture of the incidence ray cone are known, and the reflection process obeys the laws of geometric optics. In this form the problem was posed by Westcott and others. The research of these investigators has been supported by Plessey Radar, Ltd. for many years.

The problem admits a precise mathematical formulation and in this form it reduces to solving the equation

$$\frac{4P^2\det\left[\nabla_{ij}P + (P - \frac{|\nabla P|^2 + P^2}{2P})e_{ij}\right]}{(|\nabla P|^2 + P^2)^2\det(e_{ij})} = f \text{ in } \omega, \tag{8}$$

$$\text{nonlinear boundary condition on } \partial\omega \tag{9}$$

with respect to the unknown function P (>0) naturally associated with the problem; here (e_{ij}) is the matrix of the first fundamental form e of the unit spheres S^2, ∇ the gradient in the metric e, ∇_{ij}-second covariant derivatives in e, and f the prescribed power density. The condition (9) is somewhat complicated to be presented here without considerable expansion.

With the support of AFOSR we started our investigation of the problem with the radially symmetric (r.s.) case, that is, when the incidence ray cone Ω and the far field domain ω are circular, the prescribed density of reflected rays is a function of the asimutal angle only, and the reflecting surface is sought as a surface of revolution. Later we have also investigated the noradially symmetric case. We obtained the following results:

- the r.s. case is completely resolved. Simple and verifiable necessary and sufficient conditions for solubility of (8) and (9) are given and it is shown that radially symmetric solutions can be constructed explicitly whenever all of the parameters are appropriate.

- it is shown that in the nonradially symmetric case with circular incidence ray cone and far field and distribution density close (in certain norm) to a radially symmetric distribution this problem admits two classes of solutions provided the data satisfies a necessary condition expressing the energy conservation law.

We also began investigating the following inverse problem. Consider, as before, a reflector antenna system, consisting of a point light source O, a reflecting surface F and object T in space to be illuminated in this system. Under the assumptions of geometric optics theory it is desired to construct the surface F if the positions of the light source O and object T are given, and the power distribution is a prescribed in advance function on T. In addition, the aperture of the incidence ray cone is a design parameter and it is also prescribed. For this situation we developed the appropriate equation relating the phase and power distribution generated by a reflector illuminating the target. In our considerations the target is allowed to be an arbitrary surface. This is a considerable improvement of an earlier result of British researchers of Brickell and Westcott. Further, we have given general conditions for a given function to be the phase of a reflector illuminating an imbedded surface. Then we established conditions for solvability of the above inverse problem and showed that under natural restrictions one can construct a reflector surface which will illuminate a given surface of revolution, coaxial with the reflector, with prescribed in advance varying light intensity and aperture of the incidence ray cone.

II. Singular Boundary Value Problems

The nonlinear boundary value problem

$$\Delta u + f(x,u) = 0 \tag{*}$$

$$u\,|_\Gamma = \phi(x)$$

on a region Ω in R^n with boundary Γ is a problem of mathematical interest. The special case $F(x,u) = a(x)u^p, p > 0$, is known as the Emden-Fowler equation and is of considerable interest in the applied literature. The case that $p > 0$ is also of interest; $p < 0$ makes the problem singualr and is

challenging for both theoretical and numerical reasons. Many of the computational difficulties are already present in the ordinary differential equation

$$y'' + a(x)y^p = 0$$

with the usual Sturm-Liouville boundary conditions. Also of interest is the problem which yields radially symmetric solutions of (*),

$$y'' + \frac{y'}{x} + a(x)y^p = 0$$

$$y'(0) = 0,\ y(1) = 0.$$

Early in the grant generalizations of both the Emden-Fowler version and the singular version were investigated. The equation took the form

$$y'' + f(x,y) = 0$$

with general Sturm Liouville boundary conditions and with conditions on f that included either the Emden-Fowler equation or the singular equation. (The singular equation is used to model the motion of psuedo-plastic fluids. The method of proof involved the construction of iterative schemes and, in the singular case, a fixed point theorem for decreasing mappings which is, in its own right, of mathematical interest. For the non-singular case there is no numerical advantage in the iterative scheme since an integration is involved. However, in the singular case shooting methods are not appropriate since the initial value problem may not be well posed. The singularity effectively prohibits the use of finite differences schemes as well, so the iterative methods take on new importance. Moreover, iterative schemes lend themselves to parallel computation. The system was programmed and the iteration technique found to be effective. Because the programming is so simple (no large ODE solver is needed), it can be done easily on very simple personal computers. (Some of the early computations were done on an Atari to illustrate this point.) It is clear that for the partial differential equation larger machines will be needed. Since many of the computational difficulties are present in the boundary value problem for the ordinary differential equation, a parallel version has been implemented. The idea is to use one processor to collect the data and assign the task (integrations, comparisons, output) to the other processors. This is currently in progress and it is the hope that much can be learned which will be useful in the full PDE (singular) equation. The code for a program already exists and it has been obsereved experimentally that the speed of the computations increases almost linearly with respect to the number of processors available; the computations have been carried out in an Encore Mulitmax computer with twelve processors but the program will run with any number of processors.

III. Publications and Presentations of research supported by AFOSR Grant #87-0140

- paper "Radially symmetric solutions of a Monge-Ampere equation arising in a reflector mapping problem", Proceedings of the UAB Conference on Differential Equations and Mathematical Physics, ed. by I. Knowles and Y. Saito, Lecture notes in Mathematics, 125(1987), 361-374 (V. Oliker, P. Waltman)

- paper "Near radially symmetric solution of an inverse problem in geometric optics", Journal of Inverse Problems, 3(1987), 361-374 (V. Oliker)

- paper "On the numerical soluton of the equation $z_{xx}z_{yy} - (z_{xy})^2$ and its discretizations, I", Numerische Mathematik, 45(1988), 271-293 (V. Oliker, L.D. Prussner)

- an hour lecture at the Oberwolfach Conference on Variational Calculus, Germany, June 1988 (V. Oliker)

- an invited lecture at the New York Academy of Sciences, November 1988 (V. Oliker)

- paper "Singular nonlinear boundary value problems for second order ODE's", Journal of Differential Equations, v. 79, NO. 1(1989), 62-78 (J. Gatica, V. Oliker, P. Waltman)

- paper "Iterative procedures for nonlinear second order boundary value problems", Annalid di Matematica Pura ed Applicata (to appear), pp. 1-23 (J. Gatica, V. Oliker, P. Waltman)

- paper "On reconstructing a reflecting surface from the scattering data in the gemetric optics approximation", Journal of Inverse Problems, 5(1989), 51-65 (V. Oliker)

- The preceding paper appeared also in the series "Sonderforschungsbereich 123" Stochastiche Mathematische Modelle, University of Heidelberg, West Germany, preprint #466, June 1988 (V. Oliker).

- invited lecture at the AAAS Annual Meeting, New Orleans, February 1990 (V. Oliker)

- invited 1-hour address at the AMS Meeting, Fayetteville, March 1990 (V. Oliker)

- preprint "Solving stationary and nonstationary equations of Monge-Ampere type on computers with single and multiple processors", pp. 1-12, 1989 (V. Oliker, L.D. Prussner)

- preprint "Algorithm for solving the balance equation", pp. 1-17, 1990 (V. Oliker, L.D. Prussner)

- invited lecture at the Workshop on Partial Differential Equations, Georgia Institute of Technology, May, 1990

- preprint "Evolution of nonparametric surfaces with speed depending on curvature", I. -The case of Gauss curvature, pp. 1-23, 1990 (V. Oliker), Indiana Univ. J. - to appear

- the preceding paper appeared also in the series "Sonderforschungsbereich 123" Stochastiche Mathematische Modelle, University of Heidelberg, West Germany, preprint #584, July 1990 (V. Oliker)

- a 6-min. video tape was produced depicting qualitative properties of fronts propagating with speed controlled by curvature, 1990 (V. Oliker, L.D. Prussner)

- Over several years of work we developed a software package dedicated to:

  numerical solution of nonlinear equations of Monge-Ampere type;
  modeling propagation of fronts with speed controlled by the Gauss or mean curvatures;
  computing minimal surfaces;
  computing ground states of some nonlinear problem.

December, 1990